Dynamics of Continuous, Discrete and Impulsive Systems
Series **B:** *Applications & Algorithms* **14 (2007)** 641-657
 http://www.watam.org

CONTROLLABILITY CONDITIONS FOR SWITCHED LINEAR SINGULAR SYSTEMS

B. Meng[1,2] and J. F. Zhang[1]

[1]Academy of Mathematics and Systems Science
Chinese Academy of Sciences, Beijing 100080, China

[2] Beijing Institute of Control Engineering
China Academy of Space Technology, Beijing 100080, China

Abstract. The controllability problem of switched linear singular (SLS) systems is investigated in this paper. Under the regularity condition of all switching subsystems, a necessary condition and a sufficient condition for complete controllability are presented, and the conditions for complete reachability of the SLS system given in Meng and Zhang [14] are weakened. It is proved that for the SLS system under certain conditions complete controllability and complete reachability are equivalent.

Keywords. Switched system, singular system, controllability, reachability, admissible control.

AMS (MOS) subject classification: 93B05

1 Introduction

During the past few years, the study of switched systems has been revivified (see e.g. Bengea and DeCarlo [1], Cheng *et al* [4], De Santis *et al* [6], Escobar *et al* [8], Liberzon and Morse [11], Stanford and Conner [16], Vidal *et al* [22]). Various conditions and subtle results on controllability, reachability and observability etc. are presented in Ezzine and Haddad [7], Ge *et al* [10], Sun *et al* [18], Sun and Zheng [19], and Xie and Wang [24,25], respectively, for continuous-time periodic, general (non-periodic) switched control systems, and discrete-time switched control systems.

Switched linear singular (SLS) systems constitute an important class of switched systems, which arises, for example, in electrical networks and economic systems (see e.g. Bedrosian and Vlach [2], Cantó *et al* [3], Gandolfo [9], Opal and Vlach [15], Silva and de Lima [17], Tanaka [20], Tolsa and Salichs [21], Vlach *et al* [23], and the references therein). Due to the existence of switching actions, state-inconsistence phenomena often occurs. This may result in the discontinuity of network variables and in the presence of impulse voltage and currents at the switching instants. Physically, some problems like sparks and short circuits etc. may occur (Escobar *et al* [8]). For dynamic economic systems, as pointed out by Cantó *et al* [3], when the interrelationships among different industrial sectors are described, and the capital and the demand are variable depending on seasons, the system can be modelled as a periodically switched singular systems. Therefore, it is very important to analyze the behavior of the SLS systems.

It is known that by proper design, switching control can improve the transient response of the systems in some circumstances, especially when the system cannot be asymptotically stabilized by a single continuous feedback control law (Morse [12]). Actually, the problems such as how to obtain consistent initial conditions and switching transformation matrices (to express the discontinuity of the state variables at the switching instants) have already been investigated by using Laplace transformation and differential-algebraic equation (Tanaka [20], Tolsa and Salichs [21], Vlach *et al* [23]).

Due to the complexity of the structure and behavior, the analysis and synthesis of the SLS systems are more difficult. Especially, state discontinuity, impulse phenomenon and regularity should be considered at the same time. Recently, some preliminary results on SLS systems have been given in Meng and Zhang [13,14] and Yin and Zhang [26]. In Meng and Zhang [13], the case where switching law is designable is considered. Both state feedback gain matrices and switching laws are designed such that the closed-loop SLS system admissible and the states continuous. In Meng and Zhang [14], the reachability of SLS systems is studied, and a necessary condition and a sufficient condition are obtained. In Yin and Zhang [26], asymptotic properties, including complexity reduction and limit behavior, of large-scale hybrid singular systems are analyzed.

This paper is devoted to controllability problem of SLS systems. Based on the regularity condition, a necessary condition and a sufficient condition on complete controllability are given by defining an admissible control set and using the geometric approach. The sufficient condition is generalized to the complete reachability of the SLS systems and the sufficient condition given in Meng and Zhang [14] is weakened. It is proved that under certain conditions for the SLS systems complete controllability and complete reachability are equivalent.

2 Notations and preliminary results

Consider an SLS control system described by

$$E_{\sigma(t)}\dot{x} = A_{\sigma(t)}x + B_{\sigma(t)}u_{\sigma(t)}(t), \tag{1}$$

where $\sigma(t) : [0, +\infty) \to \Lambda = \{1, 2, \cdots, m\}$ is a right-continuous piecewise constant mapping; $x(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^{m_i}$, $i \in \Lambda$ are the state and input, respectively; $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m_i}$, $E_i \in \mathbb{R}^{n\times n}$, $rankE_i = r_i \leq n$, $i \in \Lambda$.

Throughout this paper, $\mathbb{C}$ denotes the set of all complex numbers; $\mathbb{Z}^+$ denotes the set of positive integers; $\mathbb{R}^n$ denotes the real n-dimensional space;

$\mathbb{R}^{n\times n}$ denotes the real $n \times n$-dimensional space; for a given vector or matrix X, X^T denotes its transpose; Λ denotes the integer set $\{1, 2, \cdots, m\}$; I_k denotes the $k\times k$ identity matrix; and for two nonnegative integers $j \geq l$ and a square matrix sequence X_k $(k = l, \cdots, j)$ with appropriate dimensions, denote the production $X_j X_{j-1} \cdots X_l$ by $\prod_{k=j}^{l} X_k$.

¿From Dai [5], it is known that a necessary and sufficient condition for the existence and uniqueness of the solution of (1) is that for all $i \in \Lambda$, (E_i, A_i) are regular. So, in this paper, we assume:

Assumption 1 *For all $i \in \Lambda$, (E_i, A_i) are regular, i.e., for every $i \in \Lambda$, there exists $s_i \in \mathbb{C}$ such that* $\det(s_iE_i - A_i) \neq 0$.

By Assumption 1, there exist nonsingular matrices $P_i \in \mathbb{R}^{n\times n}$, $Q_i \in \mathbb{R}^{n\times n}$, $i \in \Lambda$, such that

$$P_iE_iQ_i = \begin{bmatrix} I_{n_i} & 0 \\ 0 & N_i \end{bmatrix}, \quad P_iA_iQ_i = \begin{bmatrix} G_i & 0 \\ 0 & I_{n-n_i} \end{bmatrix}, \tag{2}$$

where $N_i \in \mathbb{R}^{(n-n_i)\times(n-n_i)}$ is nilpotent with nilpotent index h_i; $G_i \in \mathbb{R}^{n_i\times n_i}$. Let $Q_i = [\bar{Q}_{i1},\ \bar{Q}_{i2}]$, $Q_i^{-1} = [Q_{i1}^T\ Q_{i2}^T]^T$, and $P_iB_i = [B_{i1}^T\ B_{i2}^T]^T$ with $\bar{Q}_{i1} \in \mathbb{R}^{n\times n_i}$, $Q_{i1} \in \mathbb{R}^{n_i\times n}$, $B_{i1} \in \mathbb{R}^{n_i\times m_i}$. Then we have $Q_{i1}Q_i = [I_{n_i}\ 0]$ and $Q_{i1}\bar{Q}_{i1} = I_{n_i}$.

¿From Dai [5], we know that for any fixed regular singular subsystem

$$E_i\dot{x} = A_ix + B_iu_i,\ x(t_0) = x_0, \tag{3}$$

the solution of (3) with the initial value $x(t_0) = x_0$ and input u_i is:

$$\begin{cases} Q_{i1}x(t) = e^{G_i(t-t_0)}Q_{i1}x_0 + \displaystyle\int_{t_0}^{t} e^{G_i(t-\tau)}B_{i1}u_i(\tau)d\tau, \\ Q_{i2}x(t) = -\displaystyle\sum_{k=1}^{h_i-1} N_i^k\delta^{(k-1)}(t-t_0)\left(Q_{i2}x_0 - \sum_{r=0}^{h_i-1} N_i^r B_{i2}u_i^{(r)}(t_0^+)\right) \\ \qquad\qquad - \displaystyle\sum_{k=0}^{h_i-1} N_i^k B_{i2}u_i^{(k)}(t), \end{cases}$$

where $f^{(k)}(t)$ and $f^{(r)}(t^+)$ denote the $k-$derivative and right $r-$derivative at t of the generalized function $f(t)$ respectively.

For clarity, let $x(t; t_0, x_0, u, \sigma)$ denote the state trajectory at time t of system (1) starting from t_0 with initial value x_0, input u and switching law σ. For any given time interval $[t_1, t_2]$, suppose that $\sigma(t)$ has k switching (discontinuous) points $t_{11}, t_{12}, \cdots, t_{1k}$ $(t_1 < t_{11} < t_{12} < \cdots < t_{1k} < t_2)$, i.e. for any $t \in [t_{1j}, t_{1(j+1)})$, $\sigma(t) = \sigma(t_{1j}) \in \Lambda$, $\sigma(t_{1j}) \neq \sigma(t_{1(j+1)})$, $j = 0, 1, \cdots, k-1$, $t_{10} = t_1$, and for any $t \in [t_{1k}, t_2]$, $\sigma(t) = \sigma(t_{1k}) \in \Lambda$, then we denote this switching sequence as $\{\sigma(t_{1j}), t_{1j}\}_{j=0}^{k}$; and for any given initial value

$x(t_1) = x_0$, time interval $[t_1, t_2]$, and switching sequence $\sigma = \{\sigma(t_{1j}), t_{1j}\}_{j=0}^{k}$, we define an admissible control set $\mathcal{U}_\sigma[t_1, t_2]$ as follows:

$$\begin{aligned}\mathcal{U}_\sigma[t_1,t_2] \;=\; \Bigg\{ u:\; & u=[u_1^T,u_2^T,\cdots,\,u_m^T]^T,\; u_i\in C^{h-1}[t_1,t_2],\\ & \sum_{r=0}^{h_{\sigma(t_{1j})}-1} N_{\sigma(t_{1j})}^{r} B_{\sigma(t_{1j})2} u_{\sigma(t_{1j})}^{(r)}(t_{1j}^{+})\\ & \qquad = -Q_{\sigma(t_{1j})2}\, x(t_{1j}^{-};t_1,x_0,u,\sigma),\\ & \qquad\qquad j=0,1,\cdots,k,\; t_{10}^{-}=t_1 \Bigg\},\end{aligned}$$

where $h = \max\{h_1, h_2, \cdots, h_m\}$; $C^h[t_1, t_2]$ denotes the set of all h-differentiable functions in the time interval $[t_1, t_2]$; $u_{\sigma(t_{1j})}^{(r)}(t_{1j}^{+})$ and $x(t_{1j}^{-}; t_1, x_0, u, \sigma)$ are the right r-derivative of $u_{\sigma(t_{1j})}(t)$ at $t = t_{1j}$ and the left limit of $x(t; t_1, x_0, u, \sigma)$ at $t = t_{1j}$, respectively.

Remark 1 *It is clear that when* $\langle N_i \mid B_{i2}\rangle = \mathbb{R}^{n-n_i}$, $i = 1, 2, \cdots, m$, *there must exist piecewise continuous function* u *such that the admissible control set is nonempty. For a general SLS system the problem of existence of the admissible control is somewhat complicated since it is involved in the switching sequence and the initial state, and the switched subsystems itself as well. For instance, in the case where* $n_1 = n_2 = \cdots = n_m = 0$, $P_i = Q_i = I$, $\sum_{i=1}^{m}\langle N_i \mid B_{i2}\rangle = \mathbb{R}^n$, $\langle N_i \mid B_{i2}\rangle \cap \langle N_j \mid B_{j2}\rangle = \{0\}$, $\forall i \neq j$, *and* $\langle N_i \mid B_{i2}\rangle \neq \mathbb{R}^n$, *when the initial state belongs to the subspace* $\langle N_i \mid B_{i2}\rangle$, $i = 1, 2, \cdots, m$, *there does exist a switching sequence that ensures the existence of an admissible control. But for the other states, there do not exist such a switching sequence.*

Remark 2 *For any given switching law* σ *and time interval* $[t_1, t_2]$*, the states of system* (1) *are continuous in* $[t_1, t_2]$ *under the action of all admissible input* $u \in \mathcal{U}_\sigma[t_1, t_2]$.

Definition 1 *SLS system* (1) *is said completely controllable, if for any given initial time* $t_0 \in \mathbb{R}$ *and initial state* $x_0 \in \mathbb{R}^n$*, there exist a real number* $t_f > t_0$*, a switching law* σ *and an admissible input* $u \in \mathcal{U}_\sigma[t_0, t_f]$*, such that* $x(t_f; t_0, x_0, u, \sigma) = 0$.

Definition 2 *SLS system* (1) *is said completely reachable, if for any given initial time* $t_0 \in \mathbb{R}$ *and state* $x_f \in \mathbb{R}^n$*, there exist a real number* $t_f > t_0$*, a switching law* σ *and an admissible input* $u \in \mathcal{U}_\sigma[t_0, t_f]$*, such that* $x_f = x(t_f; t_0, 0, u, \sigma)$.

According to the definitions the controllable set and the reachable set can be defined as follows:

$$\begin{aligned}\mathcal{C} = \{x :\ \exists\, t \geq t_0,\ \sigma : [t_0, t] \to \Lambda,\ \text{and}\ u \in \mathcal{U}_\sigma[t_0, t] \\ \text{such that}\ x(t; t_0, x, u, \sigma) = 0\}\,. \\ \mathcal{R} = \{x :\ \exists\, t \geq t_0,\ \sigma : [t_0, t] \to \Lambda,\ \text{and}\ u \in \mathcal{U}_\sigma[t_0, t] \\ \text{such that}\ x = x(t; t_0, 0, u, \sigma)\}\,.\end{aligned}$$

Obviously, if there exists an $i \in \Lambda$ such that (E_i, A_i, B_i) is controllable (reachable), then by setting $\sigma(t) = i$, SLS system (1) is completely controllable (completely reachable). Thus, without loss of generality, in this paper we will study only the case where all the subsystems are not controllable (reachable), that is, $\forall i \in \Lambda$, (E_i, A_i, B_i) is not controllable (reachable).

For any given matrices $A \in \mathbb{R}^{k\times k}$, $B \in \mathbb{R}^{k\times p}$, and subspace $\mathcal{W} \subseteq \mathbb{R}^k$, denote $\mathcal{R}(B)$ the subspace spanned by the columns of B, $\langle A \mid \mathcal{W}\rangle = \sum\limits_{i=1}^{k} A^{i-1}\mathcal{W}$, and $\langle A \mid \mathcal{R}(B)\rangle$ as $\langle A \mid B\rangle$ simply. It can be shown that $\langle A \mid \mathcal{W}\rangle$ is invariant with respect to A. For convenience of citation, we introduce the following subspaces:

$$\mathcal{V}_1 = \sum_{i=1}^{m} \mathcal{C}_i,\ \mathcal{V}_k = \sum_{i=1}^{m} Q_i(\langle G_i \mid Q_{i1}\mathcal{V}_{k-1}\rangle \oplus \langle N_i \mid B_{i2}\rangle),\ k = 2, 3, \cdots, \tag{4}$$

where $\mathcal{C}_i = Q_i\langle G_i \mid B_{i1}\rangle \oplus \langle N_i \mid B_{i2}\rangle$, $i \in \Lambda$, and $\oplus$ is the direct sum in vector space. These subspaces have the following properties. The proof of the lemma is given in the Appendix.

Lemma 1 *Under Assumption 1, we have*

1. $\mathcal{V}_i \subseteq \mathcal{V}_n$, $\forall i < n$;

2. if there exists $1 < i \leq n$ *such that* $\mathcal{V}_i = \mathcal{V}_{i-1}$, *then for all* $l > i$, $\mathcal{V}_l = \mathcal{V}_i$; *and*

3. $\mathcal{V}_i = \mathcal{V}_n$, $\forall i > n$.

We know that for the regular subsystems (E_i, A_i), $i = 1, 2, \cdots, m$, the transformation matrices P_i and Q_i are not unique. It is proved that the subspaces defined in (4) are independent of the choices of matrices P_i and Q_i in Meng and Zhang [14]. A necessary condition and a sufficient condition were also given for the complete reachability of the SLS systems using the subspaces defined above as follows.

1. Under Assumption 1, if SLS system (1) is completely reachable, then $\mathcal{V}_n = \mathbb{R}^n$;

2. Under Assumption 1, if $\langle N_i \mid B_{i2}\rangle = \mathbb{R}^{n-n_i}$, $i = 1, 2, \cdots, m$ and $\mathcal{V}_n = \mathbb{R}^n$, then the SLS system (1) is completely reachable.

In the next sections of this paper, we will present a necessary condition for the controllability and a sufficient condition for the controllability and reachability based on $\mathcal{V}_n$.

3 Controllability conditions of SLS systems

In this section we will give a necessary condition and a sufficient condition for the complete controllability of SLS systems.

A necessary condition for complete controllability is summarized in Theorem 1.

Theorem 1 *Under Assumption 1, if SLS system* (1) *is completely controllable, then* $\mathcal{V}_n = \mathbb{R}^n$.

Proof. Suppose that system (1) is completely controllable. Then by Definition 1, for any given $x_0 \in \mathbb{R}^n$, there exist a switching sequence $\sigma = \{i_j, t_j\}_{j=0}^{s}$, a time $t_{s+1} > t_s$, and an admissible input $u \in \mathcal{U}_\sigma[t_0, t_{s+1}]$, such that $x(t_{s+1}; t_0, x_0, u, \sigma) = 0$.

Let $d_k = t_{k+1} - t_k$, $0 \le k \le s$. Then from the definition of $\mathcal{U}_\sigma[t_0, t_{s+1}]$, it follows that for all $k = 0, 1, \cdots, s$, $x(t_k; t_0, x_0, u, \sigma)$ are the consistent initial states of subsystems $(E_{i_k}, A_{i_k}, B_{i_k})$ under the action of u.

We now show $x(t_s; t_0, x_0, u, \sigma) \in \mathcal{V}_1$. In fact, by

$$
\begin{aligned}
0 &= x(t_{s+1}; t_0, x_0, u, \sigma) \\
&= Q_{i_s} \begin{bmatrix} e^{G_{i_s} d_s} Q_{i_s 1} x(t_s; t_0, x_0, u, \sigma) \\ + \int_{t_s}^{t_{s+1}} e^{G_{i_s}(t_{s+1}-\tau)} B_{i_s 1} u_{i_s}(\tau) d\tau \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ - \sum\limits_{k=0}^{h_{i_s}-1} N_{i_s}^k B_{i_s 2} u_{i_s}^{(k)}(t_{s+1}) \end{bmatrix},
\end{aligned}
$$

we have

$$
Q_{i_s 1} x(t_s; t_0, x_0, u, \sigma) = -\int_{t_s}^{t_{s+1}} e^{G_{i_s}(t_s-\tau)} B_{i_s 1} u_{i_s}(\tau) d\tau,
$$

which together with

$$
Q_{i_s 2} x(t_s; t_0, x_0, u, \sigma) = -\sum_{k=0}^{h_{i_s}-1} N_{i_s}^k B_{i_s 2} u_{i_s}^{(k)}(t_s),
$$

and the definitions of $Q_{i_s 1}$ and $Q_{i_s 2}$ implies that

$$
x(t_s; t_0, x_0, u, \sigma) = Q_{i_s} \begin{bmatrix} -\int_{t_s}^{t_{s+1}} e^{G_{i_s}(t_s-\tau)} B_{i_s 1} u_{i_s}(\tau) d\tau \\ -\sum\limits_{k=0}^{h_{i_s}-1} N_{i_s}^k B_{i_s 2} u_{i_s}^{(k)}(t_s) \end{bmatrix}. \tag{5}
$$

Furthermore, from Lemma A1 it follows that

$$
x(t_s; t_0, x_0, u, \sigma) = Q_{i_s} \begin{bmatrix} \sum\limits_{j=0}^{n_{i_s}-1} G_{i_s}^j B_{i_s 1} \int_{t_s}^{t_{s+1}} f_j(t_s-\tau) u_{i_s}(\tau) d\tau \\ -\sum\limits_{k=0}^{h_{i_s}-1} N_{i_s}^k B_{i_s 2} u_{i_s}^{(k)}(t_s) \end{bmatrix},
$$

where $f(\cdot)$ is a continuous function. So, by the definition of $\mathcal{V}_1$ we know that

$$x(t_s; t_0, x_0, u, \sigma) \in Q_{i_s}(\langle G_{i_s} \mid B_{i_s 1}\rangle \oplus \langle N_{i_s} \mid B_{i_s 2}\rangle) \subseteq \mathcal{V}_1. \tag{6}$$

We now investigate the property of $x(t_{s-1}; t_0, x_0, u, \sigma)$. By

$$\begin{aligned} & x(t_s; t_0, x_0, u, \sigma) \\ = \quad & Q_{i_{s-1}} \begin{bmatrix} e^{G_{i_{s-1}} d_{s-1}} Q_{i_{s-1}1} x(t_{s-1}; t_0, x_0, u, \sigma) \\ + \int_{t_{s-1}}^{t_s} e^{G_{i_{s-1}}(t_s - \tau)} B_{i_{s-1}1} u_{i_{s-1}}(\tau) d\tau \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ - \sum_{k=0}^{h_{i_{s-1}}-1} N_{i_{s-1}}^k B_{i_{s-1}2} u_{i_{s-1}}^{(k)}(t_s) \end{bmatrix} \end{aligned}$$

we have

$$\begin{aligned} Q_{i_{s-1}1} x(t_{s-1}; t_0, x_0, u, \sigma) \quad = \quad & e^{-G_{i_{s-1}} d_{s-1}} Q_{i_{s-1}1} x(t_s; t_0, x_0, u, \sigma) \\ & - \int_{t_{s-1}}^{t_s} e^{G_{i_{s-1}}(t_{s-1} - \tau)} B_{i_{s-1}1} u_{i_{s-1}}(\tau) d\tau. \end{aligned}$$

This together with

$$Q_{i_{s-1}2} x(t_{s-1}; t_0, x_0, u, \sigma) = - \sum_{k=0}^{h_{i_{s-1}}-1} N_{i_{s-1}}^k B_{i_{s-1}2} u_{i_{s-1}}^{(k)}(t_{s-1})$$

gives

$$\begin{aligned} & x(t_{s-1}; t_0, x_0, u, \sigma) \\ = \quad & Q_{i_{s-1}} \begin{bmatrix} e^{-G_{i_{s-1}} d_{s-1}} Q_{i_{s-1}1} x(t_s; t_0, x_0, u, \sigma) \\ - \int_{t_{s-1}}^{t_s} e^{G_{i_{s-1}}(t_{s-1} - \tau)} B_{i_{s-1}1} u_{i_{s-1}}(\tau) d\tau \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ - \sum_{k=0}^{h_{i_{s-1}}-1} N_{i_{s-1}}^k B_{i_{s-1}2} u_{i_{s-1}}^{(k)}(t_{s-1}) \end{bmatrix}. \end{aligned} \tag{7}$$

By (6), Lemma A1 and

$$\int_{t_{s-1}}^{t_s} e^{G_{i_{s-1}}(t_{s-1} - \tau)} B_{i_{s-1}1} u_{i_{s-1}}(\tau) d\tau \in \langle G_{i_{s-1}} \mid B_{i_{s-1}1}\rangle \subseteq \langle G_{i_{s-1}} \mid Q_{i_{s-1}1} \mathcal{V}_1\rangle$$

we have

$$x(t_{s-1}; t_0, x_0, u, \sigma) \in Q_{i_{s-1}}(\langle G_{i_{s-1}} \mid Q_{i_{s-1}1} \mathcal{V}_1\rangle \oplus \langle N_{i_{s-1}} \mid B_{i_{s-1}2}\rangle) \subseteq \mathcal{V}_2.$$

Similarly, we can show $x(t_0; t_0, x_0, u, \sigma) \in \mathcal{V}_{s+1}$. ¿From Lemma 1 we know that $\mathcal{V}_{s+1} \subseteq \mathcal{V}_n$, and so, $x_0 \in \mathcal{V}_n$. Then, by the arbitrariness of x_0 we have $\mathcal{V}_n = \mathbb{R}^n$. $\square$

Remark 3 *It is worth pointing out that the condition of Theorem 1 is not sufficient for the controllability of the SLS system (1). For the example given in Remark 1, from* $\mathcal{V}_n = \sum_{i=1}^{m} \langle N_i \mid B_{i2} \rangle$ *and* $\sum_{i=1}^{m} \langle N_i \mid B_{i2} \rangle = \mathbb{R}^n$ *it follows that the necessary condition* $\mathcal{V}_n = \mathbb{R}^n$ *is satisfied. From the detail analysis of the existence of the admissible control in Remark 1 and the definition of controllability, we know that the system is not controllable.*

Now, by using the concept of controllable set and the geometric approach we would like to give a sufficient condition for the complete controllability of system (1). For a given switching sequence $\sigma = \{i_j, t_j\}_{j=0}^{s}$, $i_j \in \Lambda$, $t_0 \leq t_1 \leq \cdots \leq t_{s+1}$, define $\mathcal{C}(t_{s+1}) = 0$. Let $d_j = t_{j+1} - t_j$, and $\mathcal{C}(t_j)$, $j = 0, 1, \cdots, s$, be the sets of states $x(t_j)$ of subsystem $(E_{i_j}, A_{i_j}, B_{i_j})$ under the action of $u_{i_j} \in \mathcal{U}_2(\mathcal{C}(t_{j+1}), [t_j, t_{j+1}])$. If $Q_{i_{k-1}2}\mathcal{C}(t_k) \subseteq \langle N_{i_{k-1}} \mid B_{i_{k-1}2} \rangle$, $k = 1, 2, \cdots, s$, then it follows from the definition of $\mathcal{C}(t_k)$, $k = 0, 1, \cdots, s$, that for any $x \in \mathcal{C}(t_0)$, there exists $u \in \mathcal{U}_\sigma[t_0, t_{s+1}]$ such that $x(t_{s+1}; t_0, x, u, \sigma) = 0$. Thus, by the definition of $\mathcal{C}$ we have

$$\mathcal{C}(t_0) \subseteq \mathcal{C}. \tag{8}$$

In the sequel, denote $H_{i_j} = \bar{Q}_{i_j 1} e^{-G_{i_j} d_j} Q_{i_j 1}$, $d_j = t_{j+1} - t_j$, $j = 0, 1, \cdots, s$. Now, we give the geometric characteristic of $\mathcal{C}(t_k)$, $k = 0, 1, \cdots, s$, in lemmas 2 and 3, whose proofs are put in the Appendix, respectively.

Lemma 2 *Under Assumption 1, if* $\langle N_i \mid B_{i2} \rangle = \mathbb{R}^{n-n_i}$, $i = 1, 2, \cdots, m$, *then*

$$\mathcal{C}(t_k) = \sum_{l=k+1}^{s} \prod_{j=k}^{l-1} H_{i_j} \mathcal{C}_{i_l} + \mathcal{C}_{i_k}.$$

A switching law $\sigma_c = (i_j, t_j)_{j=0}^{s}$, $s = lm - 1$, $l \in \mathbb{Z}^+$, $l \geq 2$, is said to be circulatory, if $i_r \in \Lambda \backslash \{i_0, i_1, \cdots, i_{r-1}\}$ and $i_{hm+r} = i_r$, $h = 1, 2, \cdots, l-1$, $r = 0, 1, \cdots, m-1$. In Lemma 3 below we will investigate the properties of the controllable set of circulatory switching laws, whose proof is given in Appendix. Furthermore, we present the geometric character for the controllable set $\mathcal{C}$ in Theorem 2.

Lemma 3 *Under Assumption 1, if* $\langle N_i \mid B_{i2} \rangle = \mathbb{R}^{n-n_i}$, $i = 1, 2, \cdots, m$, *then for any given matrix* A, *subspace* $\mathcal{F}$ *of proper dimensions, and almost all* $d_0, d_1, \cdots, d_{\tau_z - 1} \in \mathbb{R}$,

$$\dim(A\mathcal{C}(t_0) + \mathcal{F}) \geq \dim(AH_{i_{\tau_z}} \mathcal{C}(t_{\tau_z + 1}) + A\mathcal{V}_z + \mathcal{F}), \tag{9}$$

where $\bar{n} = \sum_{k=0}^{m-1} n_k$ *and* $\tau_z = \frac{\bar{n}^z + \bar{n}^{z-1} - 2}{\bar{n} - 1} m - 1$, $z \geq 1$.

Theorem 2 *Under Assumption 1, if* $\langle N_i \mid B_{i2}\rangle = \mathbb{R}^{n-n_i}$*,* $i = 1, 2, \cdots, m$*, then the controllable set* $\mathcal{C}$ *of SLS system* (1) *is a subspace, and*

$$\mathcal{C} = \mathcal{V}_n. \tag{10}$$

Proof. By the proof procedure of Theorem 1 we know that for any $x \in \mathcal{C}$, $x \in \mathcal{V}_n$. Thus,

$$\mathcal{C} \subseteq \mathcal{V}_n. \tag{11}$$

¿From (8) and Lemma 3, for almost all d_0, d_1, $\cdots$, $d_{\tau_n - 1} \in \mathbb{R}$, $\dim(\mathcal{C}) \geq \dim(\mathcal{C}(t_0)) \geq \dim(\mathcal{C}(t_{\tau_n}) + \mathcal{V}_n) \geq \dim(\mathcal{V}_n)$. This together with (11) gives (10). □

Remark 4 *Theorem 2 implies that for any* $x \in \mathcal{V}_n$*, there exist a circulatory switching law* σ_c*,* $l \geq \frac{\bar{n}^n + \bar{n}^{n-1} - 2}{\bar{n} - 1}$*,* $t_{k+1} = t_k + d_k$*,* $0 \leq k \leq s$ *(almost all* $d_0, d_1, \cdots, d_s \in \mathbb{R}$*), and an admissible control law* u_{σ_c} *such that* $x(t_{s+1}; t_0, x, u_{\sigma_c}, \sigma_c) = 0$*, i.e.,* $x \in \mathcal{C}$*.*

¿From Theorem 2 we get the sufficient condition for controllability of the SLS systems.

Theorem 3 *Under Assumption 1, if* $\mathcal{V}_n = \mathbb{R}^n$ *and* $\langle N_i \mid B_{i2}\rangle = \mathbb{R}^{n-n_i}$*,* $i = 1, 2, \cdots, m$*, then SLS system* (1) *is completely controllable.*

Remark 5 *It is worth noticing that the condition* $\langle N_i \mid B_{i2}\rangle = \mathbb{R}^{n-n_i}$ *of Theorem 3 is not necessary for the controllability of the system (1). For instance, in the case where* $n_1 = n_2 = \cdots = n_m < n$*;* $P_i = Q_i = I$*,* $i = 1, \cdots, m$*;* $B_{i2} = 0$*,* $i = 1, 2, \cdots, m-1$*;* $\langle N_m \mid B_{m2}\rangle = \mathbb{R}^{n-n_m}$*, and* $\sum_{i=1}^{m} \langle G_i \mid B_{i1}\rangle = \mathbb{R}^{n_1}$*. Similar to the analysis of Remark 9 in Meng and Zhang [14], it can be showed that the system is controllable, although* $\langle N_i \mid B_{i2}\rangle = \{0\}$*,* $i = 1, 2, \cdots, m-1$*.*

Remarks 3, 5 imply that there is a gap between the necessary condition and the sufficient condition given in Theorem 1 and Theorem 3 for the controllability of the SLS system (1). The existence of admissible inputs and the switching laws are coupled, which makes the controllability analysis of SLS systems complicated. So, it needs more efforts to find out a necessary and sufficient condition.

4 Complete reachability of SLS systems

In this section we consider the reachable set of SLS systems, and give a sufficient condition for complete reachability of SLS systems, which is weaker than that given in Meng and Zhang [14], and can be proved in a similar way showing Theorems 2 and 3.

Theorem 4 *Under Assumption 1, if* $\langle N_i \mid B_{i2}\rangle = \mathbb{R}^{n-n_i}$, $i = 1, 2, \cdots, m$, *then the reachable set* $\mathcal{R}$ *of SLS system* (1) *is a subspace, and*

$$\mathcal{R} = \mathcal{V}_n.$$

Theorem 5 *Under Assumption 1, if* $\langle N_i \mid B_{i2}\rangle = \mathbb{R}^{n-n_i}$, $i = 1, 2, \cdots, m$, *and* $\mathcal{V}_n = \mathbb{R}^n$, *then SLS system* (1) *is completely reachable.*

Remark 6 *¿From the proof of Theorem 2, one can see that both the complete controllability and the complete reachability of the SLS system* (1) *can be realized only by using a circulatory switching law.*

Remark 7 *When* $E_i = I$, $i \in \Lambda$, *the conditions given in Theorem 1, 3 and 5 degenerate to those given in Sun et al [18] for conventional switching systems.*

Remark 8 *When* $m = 1$, *the conditions given in Theorem 1, 3 and 5 degenerate to those given in Dai [5] for the controllability and reachability of singular systems.*

5 Conclusion

The controllability problem of SLS systems has been investigated in this paper under the regularity assumption of all subsystems. By using the expressions of controllable set and reachable set of the circulatory switching sequence, it has been proved that under certain conditions the controllable set and the reachable set are the same subspace, and complete controllability and complete reachability are equivalent. Based on the structure characteristic of the solution of SLS system state equation and the approach of circulatory invariant subspaces, sufficient conditions for complete controllability and complete reachability, and a necessary condition for complete controllability have been given. The conditions given here are exactly the same as those of the conventional (non-singular) switched system and normal (non-switching) singular system cases given in Sun *et al* [18] and Dai [5] when the systems degenerate to conventional systems and normal singular systems, respectively.

6 Appendix

Proof of Lemma 1. 1. For $\forall\ k > 1$, by the definition of $\mathcal{V}_{k+1}$ we have

$$\begin{aligned}\mathcal{V}_{k+1} &= \sum_{i=1}^{m} Q_i(\langle G_i \mid Q_{i1}\mathcal{V}_k\rangle \oplus \langle N_i \mid B_{i2}\rangle) \supseteq \sum_{i=1}^{m} Q_i(Q_{i1}\mathcal{V}_k \oplus \langle N_i \mid B_{i2}\rangle) \\ &\supseteq \sum_{i=1}^{m} Q_i(Q_{i1}(Q_i(\langle G_i \mid Q_{i1}\mathcal{V}_{k-1}\rangle \oplus \langle N_i \mid B_{i2}\rangle)) \oplus \langle N_i \mid B_{i2}\rangle)\end{aligned}$$

$$\supseteq \sum_{i=1}^{m} Q_i(\langle G_i \mid Q_{i1}\mathcal{V}_{k-1}\rangle \oplus \langle N_i \mid B_{i2}\rangle) = \mathcal{V}_k,$$

where the definition of $\langle \cdot \mid \cdot \rangle$, the definition of $\mathcal{V}_k$ (the ith term of $\mathcal{V}_k$ is enough), and $Q_{i1}Q_i = \left[\begin{array}{cc} I_{n_i} & 0 \end{array}\right]$ have been used for getting the first, second and third $\supseteq$, respectively.

2. If there exists $i \le n$ such that $\mathcal{V}_i = \mathcal{V}_{i-1}$, then by the definition of $\mathcal{V}_{i+1}$ we have

$$\begin{aligned} \mathcal{V}_{i+1} &= \sum_{k=1}^{m} Q_k(\langle G_k \mid Q_{k1}\mathcal{V}_i\rangle \oplus \langle N_k \mid B_{k2}\rangle) \\ &= \sum_{k=1}^{m} Q_k(\langle G_k \mid Q_{k1}\mathcal{V}_{i-1}\rangle \oplus \langle N_k \mid B_{k2}\rangle) \\ &= \mathcal{V}_i, \end{aligned}$$

that is, $\mathcal{V}_{i+1} = \mathcal{V}_i$. Similarly, for all $l > i$ we have $\mathcal{V}_l = \mathcal{V}_i$.

3. By 1 and 2 above we can obtain Item 3 directly. □

Lemma A1 (Dai [5]) *For any given matrix $A \in \mathbb{R}^{n\times n}$, there exist continuous functions $f_0(t)$, $f_1(t)$, $\cdots$, $f_{n-1}(t)$ such that*

$$e^{At} = f_0(t)I + f_1(t)A + \cdots + f_{n-1}(t)A^{n-1}.$$

In order to prove the lemma 2, we need the following lemma, which can be proved in a similar way showing Lemma 4 in Meng and Zhang [14] subject to some minor modifications.

Lemma A2 *For any given $t_2 > t_1$, matrices $A \in \mathbb{R}^{r\times r}$, $B \in \mathbb{R}^{r\times s}$ and $D \in \mathbb{R}^{(n-r)\times s}$, nilpotent matrix $N \in \mathbb{R}^{(n-r)\times(n-r)}$ with nilpotent index h, and $y \in \langle N \mid D\rangle$, denote for $i = 1, 2$,*

$$\mathcal{U}_i(y, [t_1, t_2]) = \left\{ u(t) : \; u(t) \in C^{h-1}[t_1, t_2], \; \sum_{j=0}^{h-1} N^j D u^{(j)}(t_i) = -y \right\},$$

$$\mathcal{S}_{i1} = \left\{ x_1 : \; \exists\, u(t) \in \mathcal{U}_i(y, [t_1, t_2]) \text{ such that } x_1 = \int_{t_1}^{t_2} e^{A(t_{3-i}-\tau)} B u(\tau) d\tau \right\},$$

$$\mathcal{S}_{i2} = \left\{ x_2 : \; \exists\, u(t) \in \mathcal{U}_i(y, [t_1, t_2]) \text{ such that } x_2 = -\sum_{j=0}^{h-1} N^j D u^{(j)}(t_{3-i}) \right\}.$$

Then $\mathcal{S}_{11} = \langle A \mid B\rangle = \mathcal{S}_{21}$ and $\mathcal{S}_{12} = \langle N \mid D\rangle = \mathcal{S}_{22}$.

Proof of Lemma 2. We first show that under Assumption 1, we have $\mathcal{C}(t_s)=\mathcal{C}_{i_s}$, and furthermore, if $\langle N_i \mid B_{i2}\rangle=\mathbb{R}^{n-n_i}$, $i=1,2,\cdots,m$, then

$$\mathcal{C}(t_{k-1})=H_{i_{k-1}}\mathcal{C}(t_k)+\mathcal{C}_{i_{k-1}},\quad k=1,2,\cdots,s. \tag{A1}$$

By the definition of $\mathcal{C}(t_s)$, Lemma A2 and similar to (5), we have

$$\begin{aligned}\mathcal{C}(t_s) &= \{x(t_s):\ x(t_{s+1})=0,\ u_{i_s}(t)\in\mathcal{U}_2(0,[t_s,t_{s+1}])\}\\ &= \left\{x(t_s):\ x(t_s)=Q_{i_s}\begin{bmatrix} -\int_{t_s}^{t_{s+1}} e^{G_{i_s}(t_s-\tau)}B_{i_s1}u_{i_s}(\tau)d\tau \\ -\sum\limits_{k=0}^{h_{i_s}-1} N_{i_s}^k B_{i_s2}u_{i_s}^{(k)}(t_s)\end{bmatrix},\right.\\ &\qquad\qquad \left. u_{i_s}(t)\in\mathcal{U}_2(0,[t_s,t_{s+1}])\right\}\\ &= Q_{i_s}(\langle G_{i_s}\mid B_{i_s1}\rangle\oplus\langle N_{i_s}\mid B_{i_s2}\rangle)\\ &= \mathcal{C}_{i_s}.\end{aligned}$$

Thus, $\mathcal{C}(t_s)=\mathcal{C}_{i_s}$. We now show (A1). In this case, by $Q_{i_{s-1}2}\mathcal{C}(t_s)\subseteq\langle N_{i_{s-1}}\mid B_{i_{s-1}2}\rangle$, we know that for any $x(t_s)\in\mathcal{C}(t_s)$, $Q_{i_{s-1}2}x(t_s)\in\langle N_{i_{s-1}}\mid B_{i_{s-1}2}\rangle$, and there exists consistent initial states $x(t_{s-1})$. Then by the definition of $\mathcal{C}(t_{s-1})$, similar to (7) we have

$$x(t_{s-1})=Q_{i_{s-1}}\begin{bmatrix} e^{-G_{i_{s-1}}d_{s-1}}Q_{i_{s-1}1}x(t_s)\\ -e^{-G_{i_{s-1}}d_{s-1}}\int_{t_{s-1}}^{t_s} e^{G_{i_{s-1}}(t_s-\tau)}B_{i_{s-1}1}u_{i_{s-1}}(\tau)d\tau\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ -\sum\limits_{j=0}^{h_{i_{s-1}}-1} N_{i_{s-1}}^j B_{i_{s-1}2}u_{i_{s-1}}^{(j)}(t_{s-1})\end{bmatrix}.$$

This together with the definition of $\mathcal{C}(t_{s-1})$ and Lemma A2 gives

$$\begin{aligned}\mathcal{C}(t_{s-1}) &= \{x(t_{s-1}):\ x(t_s)\in\mathcal{C}(t_s),\ u_{i_{s-1}}\in\mathcal{U}_2(Q_{i_{s-1}2}x(t_s),[t_{s-1},t_s])\}\\ &= \left\{x(t_{s-1}):\ x(t_{s-1})=Q_{i_{s-1}}\right.\\ &\qquad\times\begin{bmatrix} e^{-G_{i_{s-1}}d_{s-1}}Q_{i_{s-1}1}x(t_s)\\ -\int_{t_{s-1}}^{t_s} e^{G_{i_{s-1}}(t_{s-1}-\tau)}B_{i_{s-1}1}u_{i_{s-1}}(\tau)d\tau\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ -\sum\limits_{j=0}^{h_{i_{s-1}}-1} N_{i_{s-1}}^j B_{i_{s-1}2}u_{i_{s-1}}^{(j)}(t_{s-1})\end{bmatrix},\\ &\qquad\qquad \left. u_{i_{s-1}}(t)\in\mathcal{U}_2(Q_{i_{s-1}2}x(t_s),[t_{s-1},t_s])\right\}\end{aligned}$$

$$\begin{aligned} &= Q_{i_{s-1}}((e^{-G_{i_{s-1}}d_{s-1}}Q_{i_{s-1}1}\mathcal{C}(t_s) + \langle G_{i_{s-1}} \mid B_{i_{s-1}1}\rangle) \oplus \langle N_{i_{s-1}} \mid B_{i_{s-1}2}\rangle) \\ &= H_{i_{s-1}}\mathcal{C}(t_s) + \mathcal{C}_{i_{s-1}}. \end{aligned}$$

Thus, (A1) is true for $k = s$. Similarly, we can prove that (A1) holds for $k = 1, 2, \cdots, s-1$.

Using (A1) with the initial value $\mathcal{C}(t_s) = \mathcal{C}_{i_s}$, we can get the conclusion iteratively. □

In order to prove Lemma 3, we need the following lemmas, where the proof of Lemma A4 is given in Sun *et al* [18] when $n = r$, and can similarly be obtained when $n \neq r$.

Lemma A3 (Sun *et al* [18]) *For any given matrix $A \in \mathbb{R}^{p\times p}$, subspace $\mathcal{B} \subseteq \mathbb{R}^p$ and almost all $t_1, t_2, \cdots, t_p \in \mathbb{R}$, we have*

$$e^{At_1}\mathcal{B} + e^{At_2}\mathcal{B} + \cdots + e^{At_p}\mathcal{B} = \langle A \mid \mathcal{B}\rangle.$$

Lemma A4 *For any given matrices $A_1 \in \mathbb{R}^{n\times r}$, $A_2 \in \mathbb{R}^{r\times r}$, $B_1 \in \mathbb{R}^{r\times p_1}$, $B_2 \in \mathbb{R}^{n\times p_2}$ and almost all $t \in \mathbb{R}$, we have*

$$rank[A_1e^{A_2t}B_1,\ B_2] \geq rank[A_1B_1,\ B_2].$$

Lemma A5 *For any given matrix $A \in \mathbb{R}^{n\times n}$, subspaces $\mathcal{B} \subseteq \mathbb{R}^n$ and $\mathcal{W} \subseteq \mathbb{R}^n$, if $Q_{i2}\mathcal{W} \subseteq \langle N_i \mid B_{i2}\rangle$, $i \in \Lambda$, then for almost all $d \in \mathbb{R}$,*

$$\dim(A(\bar{Q}_{i_1}e^{-G_id}Q_{i_1}\mathcal{W} + \mathcal{V}_j) + \mathcal{B}) \geq \dim(A(\mathcal{W} + \mathcal{V}_j) + \mathcal{B}),\ \forall j = 1, 2, \cdots, n.$$

Proof. From the definition of Q_{i1} and Q_{i2} it follows that $Q_i^{-1}\mathcal{W} \subseteq Q_{i1}\mathcal{W} \oplus Q_{i2}\mathcal{W}$. This gives

$$\begin{aligned} A\mathcal{W} + \mathcal{B} &= AQ_iQ_i^{-1}\mathcal{W} + \mathcal{B} \ \subseteq\ AQ_i(Q_{i1}\mathcal{W} \oplus Q_{i2}\mathcal{W}) + \mathcal{B} \\ &\subseteq AQ_i(Q_{i1}\mathcal{W} \oplus \langle N_i \mid B_{i2}\rangle) + \mathcal{B}. \end{aligned} \tag{A2}$$

This together with Lemma A4 and $\bar{Q}_{i2}\langle N_i \mid B_{i2}\rangle \subseteq \mathcal{V}_j$, $j = 1, 2, \cdots, n$, implies that

$$\begin{aligned} & \dim(A(\bar{Q}_{i1}e^{-G_id}Q_{i1}\mathcal{W} + \mathcal{V}_j) + \mathcal{B}) \\ \geq\ & \dim(A(\bar{Q}_{i1}Q_{i1}\mathcal{W} + \mathcal{V}_j) + \mathcal{B}) \\ =\ & \dim(A(\bar{Q}_{i1}Q_{i1}\mathcal{W} + \bar{Q}_{i2}\langle N_i \mid B_{i2}\rangle + \mathcal{V}_j) + \mathcal{B}) \\ =\ & \dim(AQ_i(Q_{i1}\mathcal{W} \oplus \langle N_i \mid B_{i2}\rangle) + A\mathcal{V}_j + \mathcal{B}) \\ \geq\ & \dim(A(\mathcal{W} + \mathcal{V}_j) + \mathcal{B}),\ \forall j = 1, 2, \cdots, n. \end{aligned}$$

We complete the proof. □

Lemma A6 *Under Assumption 1, if $\langle N_i \mid B_{i2}\rangle = \mathbb{R}^{n-n_i}$, $i = 1, 2, \cdots, m$, then for any given matrix $A \in \mathbb{R}^{n\times n}$, subspace $\mathcal{F} \subseteq \mathbb{R}^n$ and almost all $d_{k-1} \in \mathbb{R}$,*

$$\dim(A\mathcal{C}(t_{k-1}) + \mathcal{F}) \geq \dim(A\mathcal{C}(t_k) + A\mathcal{C}_{i_{k-1}} + \mathcal{F}),\ k = 1, 2, \cdots, s.$$

Proof. From Lemma A4, (A1) and $A\bar{Q}_{i_{k-1}2}\langle N_{i_{k-1}} \mid B_{i_{k-1}2}\rangle \subseteq A\mathcal{C}_{i_{k-1}}$ it follows that for almost all $d_{k-1} \in \mathbb{R}$, $k = 1, 2, \cdots, s$,

$$\begin{aligned} & \dim(A\mathcal{C}(t_{k-1}) + \mathcal{F}) \\ = \ & \dim(AH_{i_{k-1}}\mathcal{C}(t_k) + A\mathcal{C}_{i_{k-1}} + \mathcal{F}) \\ \geq \ & \dim(A\bar{Q}_{i_{k-1}1}Q_{i_{k-1}1}\mathcal{C}(t_k) + A\mathcal{C}_{i_{k-1}} + \mathcal{F}) \\ = \ & \dim(AQ_{i_{k-1}}(Q_{i_{k-1}1}\mathcal{C}(t_k) \oplus \langle N_{i_{k-1}} \mid B_{i_{k-1}2}\rangle) + A\mathcal{C}_{i_{k-1}} + \mathcal{F}). \end{aligned}$$

This together with (A2) and $Q_{i_{k-1}2}\mathcal{C}(t_k) \subseteq \langle N_{i_{k-1}} \mid B_{i_{k-1}2}\rangle$, $k = 1, 2, \cdots, s$, results in

$$\dim(A\mathcal{C}(t_{k-1}) + \mathcal{F}) \geq \dim(A\mathcal{C}(t_k) + A\mathcal{C}_{i_{k-1}} + \mathcal{F}), \ k = 1, 2, \cdots, s.$$

□

Proof of Lemma 3. The proof is given by induction. From Lemma A6 it follows that for almost all $d_0, d_1, \cdots, d_{\tau_1 - 1} \in \mathbb{R}$,

$$\begin{aligned} \dim(A\mathcal{C}(t_0) + \mathcal{F}) \ & \geq \ \dim(A\mathcal{C}(t_1) + A\mathcal{C}_{i_0} + \mathcal{F}) \\ & \geq \ \dim\Big(A\mathcal{C}(t_2) + \sum_{k=1}^{2} A\mathcal{C}_{i_{k-1}} + \mathcal{F}\Big) \\ & \geq \ \cdots \\ & \geq \ \dim\Big(A\mathcal{C}(t_{\tau_1}) + \sum_{k=1}^{m-1} A\mathcal{C}_{i_{k-1}} + \mathcal{F}\Big) \\ & = \ \dim(AH_{i_{\tau_1}}\mathcal{C}(t_{\tau_1+1}) + A\mathcal{V}_1 + \mathcal{F}), \end{aligned} \tag{A3}$$

where (A1), the definition of $\mathcal{V}_1$ and the circulatory property of σ_c have been used for the last equality, i.e. the lemma holds for $z = 1$.

Suppose that (9) holds for $z = p$, i.e. for any given matrix A, subspace $\mathcal{F}$ of proper dimensions, and almost all $d_0, d_1, \cdots, d_{\tau_p - 1} \in \mathbb{R}$,

$$\dim(A\mathcal{C}(t_0) + \mathcal{F}) \geq \dim(AH_{i_{\tau_p}}\mathcal{C}(t_{\tau_p+1}) + A\mathcal{V}_p + \mathcal{F}). \tag{A4}$$

We now consider the case of $z = p + 1$. From (A4) it follows that for almost all $d_m, d_{m+1}, \cdots, d_{\tau_p + m - 1} \in \mathbb{R}$,

$$\begin{aligned} & \dim(AH_{i_{m-1}}\mathcal{C}(t_m) + A\mathcal{V}_1 + \mathcal{F}) \\ \geq \ & \dim\Big(A\bar{Q}_{i_{m-1}1}\prod_{k=0}^{1} e^{-G_{i_{k(\tau_p+1)+m-1}}d_{k(\tau_p+1)+m-1}}Q_{i_{\tau_p+m}1}\mathcal{C}(t_{\tau_p+m+1}) \\ & \quad + AH_{i_{m-1}}\mathcal{V}_p + A\mathcal{V}_1 + \mathcal{F}\Big) \end{aligned} \tag{A5}$$

where $Q_{i_{m-1}1}\bar{Q}_{i_{m-1}1} = I_{r_1}$ and $\bar{Q}_{i_{m-1}1} = \bar{Q}_{i_{\tau_p+m}1}$ have been used, and $r_k = \sum\limits_{j=1}^{k} n_{i_{m+j-2}}$, $k = 1, 2, \cdots, m$.

Continuing the above analysis procedure, we can see that for almost all $d_{\tau_p+m+1}, \cdots, d_{2\tau_p+m}, \cdots, d_{(r_1-1)(\tau_p+1)+m}, \cdots, d_{r_1(\tau_p+1)+m-2}$,

$$\begin{aligned}
&\dim\Big(A\bar{Q}_{i_{m-1}1}\prod_{k=0}^{1} e^{-G_{i_{k(\tau_p+1)+m-1}}d_{k(\tau_p+1)+m-1}}Q_{i_{\tau_p+m}1}\mathcal{C}(t_{\tau_p+m+1})\\
&\qquad + AH_{i_{m-1}}\mathcal{V}_p + A\mathcal{V}_1 + \mathcal{F}\Big)\\
\geq\ &\dim\Big(A\bar{Q}_{i_{m-1}1}\prod_{k=0}^{2} e^{-G_{i_{k(\tau_p+1)+m-1}}d_{k(\tau_p+1)+m-1}}Q_{i_{2\tau_p+m+1}1}\mathcal{C}(t_{2\tau_p+m+2})\\
&\qquad + A\bar{Q}_{i_{m-1}1}\sum_{j=0}^{1}\prod_{k=0}^{j} e^{-G_{i_{k(\tau_p+1)+m-1}}d_{k(\tau_p+1)+m-1}}Q_{i_{m-1}1}\mathcal{V}_p\\
&\qquad + A\mathcal{V}_1 + \mathcal{F}\Big)\\
\geq\ &\cdots\\
\geq\ &\dim\Big(A\bar{Q}_{i_{m-1}1}\prod_{k=0}^{r_1} e^{-G_{i_{k(\tau_p+1)+m-1}}d_{k(\tau_p+1)+m-1}}Q_{i_{r_1(\tau_p+1)+m-1}1}\\
&\qquad \times\mathcal{C}(t_{r_1(\tau_p+1)+m}) + A\bar{Q}_{i_{m-1}1}\sum_{j=0}^{r_1-1}\prod_{k=0}^{j} e^{-G_{i_{k(\tau_p+1)+m-1}}d_{k(\tau_p+1)+m-1}}\\
&\qquad \times Q_{i_{m-1}1}\mathcal{V}_p + A\mathcal{V}_1 + \mathcal{F}\Big) \qquad\qquad (A6)
\end{aligned}$$

where the circulatory property of σ_c has been used.

Notice from Lemma A3 that for almost all d_{m-1}, d_{τ_p+m}, $\cdots$, $d_{(r_1-1)(\tau_p+1)+m-1} \in \mathbb{R}$,

$$\sum_{j=0}^{r_1-1} e^{-G_{i_{m-1}}\left(\sum_{k=0}^{j} d_{k(\tau_p+1)+m-1}\right)}Q_{i_{m-1}1}\mathcal{V}_p = \langle G_{i_{m-1}} \mid Q_{i_{m-1}1}\mathcal{V}_p\rangle.$$

Then by (A5)-(A6), Lemma A4 and Lemma A5 we have that for almost all $d_{m-1}, \cdots, d_{r_1(\tau_p+1)+m-1} \in \mathbb{R}$,

$$\begin{aligned}
&\dim(AH_{i_{m-1}}\mathcal{C}(t_m) + A\mathcal{V}_1 + \mathcal{F})\\
\geq\ &\dim\Big(A\bar{Q}_{i_{m-1}1}\prod_{k=0}^{r_1} e^{-G_{i_{k(\tau_p+1)+m-1}}d_{k(\tau_p+1)+m-1}}Q_{i_{r_1(\tau_p+1)+m-1}1}\\
&\qquad \times\mathcal{C}(t_{r_1(\tau_p+1)+m}) + A\bar{Q}_{i_{m-1}1}\langle G_{i_{m-1}} \mid Q_{i_{m-1}1}\mathcal{V}_p\rangle + A\mathcal{V}_1 + \mathcal{F}\Big)\\
\geq\ &\dim\Big(A\mathcal{C}(t_{r_1(\tau_p+1)+m}) + A\bar{Q}_{i_{m-1}1}\langle G_{i_{m-1}} \mid Q_{i_{m-1}1}\mathcal{V}_p\rangle + A\mathcal{V}_1 + \mathcal{F}\Big)\\
=\ &\dim\Big(AH_{i_{r_1(\tau_p+1)+m}}\mathcal{C}(t_{r_1(\tau_p+1)+m+1}) + A\bar{Q}_{i_{m-1}1}\langle G_{i_{m-1}} \mid Q_{i_{m-1}1}\mathcal{V}_p\rangle\\
&\qquad + A\mathcal{V}_1 + \mathcal{F}\Big). \qquad\qquad (A7)
\end{aligned}$$

Repeating the above analysis procedures, we have that for almost all $d_{r_1(\tau_p+1)+m}, \cdots, d_{r_2(\tau_p+1)+m}, \cdots, d_{r_{m-1}(\tau_p+1)+2m-2}, \cdots, d_{r_m(\tau_p+1)+2m-2}$,

$$\begin{aligned}
&\dim\Big(AH_{i_{r_1(\tau_p+1)+m}}\mathcal{C}(t_{r_1(\tau_p+1)+m+1}) + A\bar{Q}_{i_{m-1}1}\langle G_{i_{m-1}} \mid Q_{i_{m-1}1}\mathcal{V}_p\rangle \\
&\qquad + A\mathcal{V}_1 + \mathcal{F}\Big) \\
\geq\ &\dim\Big(AH_{i_{r_2(\tau_p+1)+m+1}}\mathcal{C}(t_{r_2(\tau_p+1)+m+2}) + A\sum_{j=-1}^{0}\bar{Q}_{i_{m+j}1} \\
&\qquad \times \langle G_{i_{m+j}} \mid Q_{i_{m+j}1}\mathcal{V}_p\rangle + A\mathcal{V}_1 + \mathcal{F}\Big) \\
\geq\ &\cdots \\
\geq\ &\dim\Big(AH_{i_{r_m(\tau_p+1)+2m-1}}\mathcal{C}(t_{r_m(\tau_p+1)+2m}) + A\sum_{j=-1}^{m-2}\bar{Q}_{i_{m+j}1} \\
&\qquad \times \langle G_{i_{m+j}} \mid Q_{i_{m+j}1}\mathcal{V}_p\rangle + A\mathcal{V}_1 + \mathcal{F}\Big) \\
=\ &\dim\Big(AH_{i_{r_m(\tau_p+1)+2m-1}}\mathcal{C}(t_{r_m(\tau_p+1)+2m}) + A\mathcal{V}_{p+1} + \mathcal{F}\Big) \\
=\ &\dim\Big(AH_{i_{\bar{n}(\tau_p+1)+2m-1}}\mathcal{C}(t_{\bar{n}(\tau_p+1)+2m}) + A\mathcal{V}_{p+1} + \mathcal{F}\Big) \\
=\ &\dim\Big(AH_{i_{\tau_{p+1}}}\mathcal{C}(t_{\tau_{p+1}+1}) + A\mathcal{V}_{p+1} + \mathcal{F}\Big)
\end{aligned}$$

This together with (A3) and (A7) implies that (9) holds for $z = p+1$. Thus, by the induction principle we complete the proof. □

7 Acknowledgements

This work was supported in part by the National Natural Science Foundation of China under Grants 90405017, 60274021, 60334040, and in part by the 973 program of China under Grant 2002CB312205.

8 References

[1] S. C. Bengea and R. A. DeCarlo, Optimal control of switching systems, *Automatica*, **41**, (2005) 11–27.

[2] D. Bedrosian and J. Vlach, Time-domain analysis of networks with internally controlled switches, *IEEE Trans. Circuits Syst. I*, **39**, (1992) 199–212.

[3] B. Cantó, C. Coll and E. Sánchez, Positive N-periodic descriptor control systems, *Syst. Control Lett.*, **53**, (2004) 407-414.

[4] D. Cheng, L. Guo and J. Huang, On quadratic Lyapunov functions, *IEEE Trans. Autom. Control*, **48**, (2003) 885–890.

[5] L. Dai, Singular Control Systems, Lecture Notes in Control and Information Sciences, vol. 118, Springer-Verlag, Berlin, 1989.

[6] E. De Santis, M. D. Di Benedetto and G. Pola, On observability and detectability of continuous-time linear switching systems, *Proc. of 42nd Conference Decision and Control*, Maui, HI, Dec. 2003, pp.5777–5782.

[7] J. Ezzine and H. Haddad, Controllability and observability of hybrid systems, *Int. J. Control*, **49**, (1989) 2045–2055.

[8] G. Escobar, A. J. van der Schaft and R. Ortega, A hamiltonian viewpoint in the modelling of switching power converters, *Automatica*, **35**, (1999) 445–452.

[9] G. Gandolfo, Economic Dynamics, Springer, New York, 1996.

[10] S. S. Ge, Z. Sun and T. H. Lee, Reachability and controllability of switched linear discrete-time systems, *IEEE Trans. Autom. Control*, **46**, (2001) 1437–1441.

[11] D. Liberzon and A. S. Morse, Basic problems in stability and design of switched systems, *IEEE Control Syst. Mag.*, **19**, (1999) 59–70.

[12] A. S. Morse, Supervisory control of families of linear set-point controllers, part 1: exact matching, *IEEE Trans. Autom. Control*, **41**, (1996) 1413–1431.

[13] B. Meng and J. F. Zhang, Admissible switched feedback control design of singular systems, *Proc. of 23rd Chinese Control Conference*, Wuxi, 2004, pp.1615–1619.

[14] B. Meng and J. F. Zhang, Reachability conditions for switched singular systems, *IEEE Trans. Autom. Control*, **51**, (2006).

[15] A. Opal and J. Vlach, Consistent initial conditions of linear switched networks, *IEEE Trans. Circuits syst. I*, **37**, (1990) 364–372.

[16] D. P. Stanford and L. T. Conner, Controllability and stabilizability in multi-pair systems, *SIAM J. Control Opt.*, **18**, (1980) 488–497.

[17] M. S. Silva and T. P. de Lima, Looking for nonnegative solutions of a Leontief dynamic model, *Lin. Alg. Appl.*, **364**, (2003) 281–316.

[18] Z. Sun, S. S. Ge and T. H. Lee, Controllability and reachability criteria for switched linear systems, *Automatica*, **38**, (2002) 775–786.

[19] Z. Sun and D. Zheng, On reachability and stabilization of switched linear systems, *IEEE Trans. Autom. Control*, **46**, (2001) 291–295.

[20] M. Tanaka, Formulations for switching transformation matrices of large switched networks, *Proc. of IEEE International Symposium on Circuits and Systems*, Espoo, Finland, 1988, pp.1487–1490.

[21] J. Tolsa and M. Salichs, Analysis of linear networks with inconsistent initial conditions, *IEEE Trans. Circuits Syst. I*, **40**, (1993) 885–894.

[22] R. Vidal, A. Chiuso, S. Soatto and S. Sastry, Observability of linear hybrid systems, *Hybrid Systems: Computation and Control*, (2003) 526–539 .

[23] J. Vlach, J. Wojciechowski and A. Opal, Analysis of nonlinear networks with inconsistent initial conditions, *IEEE Trans. Circuits Syst. I*, **42**, (1995) 195–200.

[24] G. Xie and L. Wang, Controllability and stabilizability of switched linear-systems, *Syst. Control Lett.*, **48**, (2003) 135–155.

[25] G. Xie and L. Wang, Reachability realization and stabilizability of switched linear discrete-time systems, *J. Math. Anal. Appl.*, **280**, (2003) 209–220.

[26] G. Yin and J. F. Zhang, Hybrid singular systems of differential equations, *Science in China (Series F)*, **45**, (2002) 241–258.

email: journal@monotone.uwaterloo.ca
http://monotone.uwaterloo.ca/∼journal/
Received November 2005; revised September 2006.